Discussion Paper Series
RIEB
Kobe University

DP2011-23

Existence and Uniqueness of a Fixed Point for the Bellman Operator in Deterministic Dynamic Programming

Takashi KAMIHIGASHI

July 10, 2011

Research Institute for Economics and Business Administration
Kobe University
2-1 Rokkodai, Nada, Kobe 657-8501 JAPAN

Existence and Uniqueness of a Fixed Point for the Bellman Operator in Deterministic Dynamic Programming*

Takashi Kamihigashi†

July 10, 2011

Abstract

We study existence and uniqueness of a fixed point for the Bellman operator in deterministic dynamic programming. We show that removing many of the assumptions of the theorem on the Bellman operator recently shown by Martins-da-Rocha and Vailakis ("Existence and Uniqueness of a Fixed Point for Local Contractions," *Econometrica* 78, 1127–1141, 2010) does not affect its main conclusions concerning the existence and uniqueness of a fixed point. Under considerably weaker versions of the remaining assumptions, we also show that the value function can be computed by value iteration with an appropriate initial function.

Keywords: Dynamic programming, Bellman operator, value function, fixed point.

---

*This is an extensively revised version of the paper presented as "The Bellman Operator as a Monotone Map" at the 11th SAET Conferance in Faro, 2011. I would like to thank all participants for helpful comments and discussions. I am grateful to Larry Blume for his encouraging comments and suggestions. Financial support from the Japan Society for the Promotion of Science (KAKENHI No. 22243024) is gratefully acknowledged.

†RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 JAPAN. Email: tkamihig@rieb.kobe-u.ac.jp. Tel/Fax: +81-78-803-7015.

1 Introduction

Building on work by Rinćon-Zapatero and Rodríguez-Palmero (2003, 2007, 2009), Martins-da-Rocha and Vailakis (2008, 2010) recently established one of the most general results on existence and uniqueness of a solution to the Bellman equation in dynamic programming—or a fixed point of the Bellman operator—currently available in the literature. Their theorem is particularly useful for treating dynamic optimization problems in which the return function is unbounded below. Among the assumptions of their theorem are the following:

(a) The state space is $\mathbb{R}^n_+$.

(b) The feasibility correspondence is continuous and compact-valued.

(c) The return function is continuous.

(d) There is a continuous function of the current state such that a return of $-\infty$ can be avoided by choosing the next state according to this function unless the current state is 0.

Using these and other assumptions, Martins-da-Rocha and Vailakis (2008, 2010) showed that the Bellman operator is a local contraction to apply their general fixed point theorem on local contractions. This is an extremely useful approach that yields not only the existence and uniqueness of a fixed point within a restricted domain of the Bellman operator, but also the convergence of value iteration from an arbitrary initial function in that domain.

In this paper, we show that the assumptions listed above are in fact irrelevant to the existence and uniqueness of a fixed point for the Bellman operator given the other assumptions of Martins-da-Rocha and Vailakis's (2010) theorem; the assumptions listed above are needed only for the convergence result and the continuity of the value function. More specifically, under considerably weaker versions of the remaining assumptions of Martins-da-Rocha and Vailakis's (2010) theorem, we show that the Bellman operator has a unique fixed point, which is the value function, in a certain order interval of functions; we also show that the value function can be computed by value iteration starting from the lower boundary of the order interval.

This paper can also be regarded as an extension of Stokey and Lucas's (1989, Sec. 4.1) analysis in their most abstract setup. Our framework is more general in that the return function is allowed to be $-\infty$, which is also the case with Martins-da-Rocha and Vailakis's (2010) theorem, and in that the objective function can be defined as the limit inferior or superior of the finite sums of discounted returns, which always exists unlike the limit. Although the uniqueness part of our result can be shown by extending some of Stokey and Lucas's (1989, Theorem 4.3) arguments,[1] the existence and convergence parts require an additional tool.[2] In the case of Martins-da-Rocha and Vailakis (2008, 2010), it is their fixed point theorem on local contractions for both existence and convergence (as well as uniqueness). In our case, we exploit the monotonicity of the Bellman operator, which allows us to apply the Knaster-Tarski fixed point theorem (e.g., Aliprantis and Border, 1999) for existence, and to develop some monotonicity-based arguments for convergence.[3]

The rest of the paper is organized as follows. In the next section, we present our framework and state our main result. In Section 3, we discuss this result in comparison with Martins-da-Rocha and Vailakis's (2010) theorem on the Bellman operator. We show that our assumptions are considerably weaker than theirs by identifying a small subset of their assumptions that is sufficient to ensure the existence and uniqueness of a fixed point.

In Appendix A, we prove our main result. In Appendix B, we present two examples. The first is trivial but has a continuum of fixed points, illustrating the importance of restricting the domain of the Bellman operator. In the second example, we show that value iteration starting from the zero function fails to converge to the value function, even under the hypotheses of our main result. This illustrates the importance of an appropriate initial function.

2 The Main Result

Let X be a set. Let Γ be a nonempty-valued correspondence from X to X. Let D be the graph of Γ:

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\}. \tag{2.1}$$

[1] We do not entirely follow this approach since we prove the uniqueness part along with the existence and convergence parts.

[2] Stokey and Lucas's (1989, Theorem 4.2) result that the value function is a fixed point of the Bellman operator is not directly applicable here, since a priori the value function need not lie in the order interval of functions that we consider.

[3] See Le Van and Morhaim (2002) for an extension of Stokey and Lucas's (1989, Sec. 4.1) analysis based on continuity assumptions like (a)–(c) above.

Let $u : D \to [-\infty, \infty)$. In the optimization problem introduced below, X is the state space, Γ is the feasibility correspondence, u is the return function, and D is the domain of u.

Let Π and $\Pi(x_0)$ denote the set of feasible paths and that of feasible paths from x_0, respectively:

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : \forall t \in \mathbb{Z}_+, x_{t+1} \in \Gamma(x_t)\}. \tag{2.2}$$

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\}, \qquad x_0 \in X. \tag{2.3}$$

Let $\beta \geq 0$. Given $x_0 \in X$, consider the following optimization problem:

$$\sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \operatorname*{L}_{T\uparrow\infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}), \tag{2.4}$$

where $\mathrm{L} \in \{\underline{\lim}, \overline{\lim}, \lim\}$ with $\underline{\lim} = \liminf$ and $\overline{\lim} = \limsup$. Since $u(x, y) < \infty$ for all $(x, y) \in D$, the objective function is well-defined for any feasible path provided that $\mathrm{L} = \underline{\lim}$ or $\overline{\lim}$. If $\mathrm{L} = \lim$, we assume that $\lim_{T\uparrow\infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1})$ exists in $\overline{\mathbb{R}}$ for each feasible path $\{x_t\}_{t=0}^{\infty}$.[4]

For $\{x_t\}_{t=0}^{\infty} \in \Pi$, we define

$$S(\{x_t\}_{t=0}^{\infty}) = \operatorname*{L}_{T\uparrow\infty} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}). \tag{2.5}$$

The value function $v^* : X \to \overline{\mathbb{R}}$ is defined by

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} S(\{x_t\}_{t=0}^{\infty}), \qquad x_0 \in X. \tag{2.6}$$

Note that $v^*(x_0)$ remains unchanged if $\Pi(x_0)$ is replaced by $\Pi^0(x_0)$,[5] where

$$\Pi^0 = \{\{x_t\}_{t=0}^{\infty} \in \Pi : S(\{x_t\}_{t=0}^{\infty}) > -\infty\}, \tag{2.7}$$

$$\Pi^0(x_0) = \{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0) : \{x_t\}_{t=0}^{\infty} \in \Pi^0\}, \qquad x_0 \in X. \tag{2.8}$$

Let V be the set of functions from X to $[-\infty, \infty)$. The Bellman operator B on V is defined by

$$(Bv)(x) = \sup_{y \in \Gamma(x)} \{u(x, y) + \beta v(y)\}, \qquad x \in X, v \in V. \tag{2.9}$$

[4] This is assumed in Martins-da-Rocha and Vailakis's (2010, p. 1131) definition of the value function v^*.

[5] We follow the convention that $\sup \emptyset = -\infty$.

Given $v \in V$, it need not be the case that $Bv \in V$. A fixed point of B is a function $v \in V$ such that $Bv = v$.

Let $v, w \in V$. We write '$v \leq w$' if $v(x) \leq w(x)$ for all $x \in X$, and '$v < w$' if $v \leq w$ and $v \neq w$. It is easy to see that B is monotone, or order-preserving:

$$v \leq w \quad \Rightarrow \quad Bv \leq Bw. \tag{2.10}$$

If $v \leq w$, we define the order interval $[v, w]$ by

$$[v, w] = \{f \in V : v \leq f \leq w\}. \tag{2.11}$$

We are ready to state the main result of the paper:

Theorem 2.1. *Suppose that there exist* $\underline{v}, \overline{v} \in V$ *such that*

$$\underline{v} \leq \overline{v}, \tag{2.12}$$

$$B\underline{v} \geq \underline{v}, \tag{2.13}$$

$$B\overline{v} \leq \overline{v}, \tag{2.14}$$

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi^0, \quad \varliminf_{t \uparrow \infty} \beta^t \underline{v}(x_t) \geq 0, \tag{2.15}$$

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi, \quad \varlimsup_{t \uparrow \infty} \beta^t \overline{v}(x_t) \leq 0. \tag{2.16}$$

Then the following conclusions hold:
(a) *The Bellman operator B has a unique fixed point in $[\underline{v}, \overline{v}]$.*
(b) *The unique fixed point of B in $[\underline{v}, \overline{v}]$ is the value function v^*.*
(c) *The sequence $\{B^n \underline{v}\}_{n=1}^{\infty}$ converges to v^* pointwise.*

Proof. See Appendix A. $\square$

Rinćon-Zapatero and Rodríguez-Palmero (2003) offer several nontrivial examples satisfying (2.12)–(2.16). Even under (2.12)–(2.14), if (2.15) and (2.16) are violated, the Bellman operator B can have multiple fixed points in $[\underline{v}, \overline{v}]$; see Appendix B for an example. In the next section, we discuss the above result in comparison with Martins-da-Rocha and Vailakis's (2010) theorem; here we comment on conclusion (c) and one implication of uniqueness.

In conclusion (c), we have convergence to v^* only from $\underline{v}$, because $\{B^n \underline{v}\}_{n=1}^{\infty}$ is the only sequence known to be increasing. Since this sequence is increasing, it converges pointwise and the limit can be expressed as the supremum of the sequence. This allows us to interchange the order of this supremum and another supremum to show that the limit is the value function v^*. This argument however does not apply to the decreasing sequence $\{B^n\overline{v}\}_{n=1}^{\infty}$, which also converges pointwise. In Appendix B, we present an example satisfying (2.12)–(2.16) in which $\lim_{n\uparrow\infty} B^n\overline{v} > v^*$.

An interesting implication of the uniqueness of the fixed point is that since the Bellman operator B does not depend on the definition of L, the value function does not depend on L either; i.e., for any $x_0 \in X$, we have

$$\sup_{\{x_t\}_{t=1}^{\infty}\in\Pi(x_0)} \varliminf_{T\uparrow\infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) = \sup_{\{x_t\}_{t=1}^{\infty}\in\Pi(x_0)} \varlimsup_{T\uparrow\infty} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}). \tag{2.17}$$

This is because under (2.12)–(2.16), both sides equal the unique fixed point of B, which does not depend on the definition of L.

3 Martins-da-Rocha and Vailakis's Theorem

Martins-da-Rocha and Vailakis (2010, Sec. 3) consider the Bellman operator under the additional assumptions listed below:

DP0. (i) $X = \mathbb{R}_+^m$ for some $m \in \mathbb{N}$. (ii) $\beta \in (0,1)$. (iii) $\mathrm{L} = \lim$.

DP1. The feasibility correspondence Γ is continuous and compact-valued.

DP2. The return function u is continuous on D.

Let $C(X, [-\infty,\infty))$ be the set of continuous functions from X to $[-\infty,\infty)$. Under DP0(i), we define

$$X^* = X \setminus \{0\}, \tag{3.1}$$

$$C^*(X) = \{v \in C(X, [-\infty,\infty)) : \forall x \in X^*, v(x) > -\infty\}. \tag{3.2}$$

DP3. There exists a continuous function $q : X^* \to X^*$ such that

$$\forall x \in X^*, \qquad (x, q(x)) \in D, \quad u(x, q(x)) > -\infty. \tag{3.3}$$

DP4. (i) There exist $\underline{v}, \overline{v} \in C^*(X)$ satisfying (2.12)–(2.14). (ii) We have

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi^0, \quad \lim_{t\uparrow\infty} \beta^t \underline{v}(x_t) = 0, \tag{3.4}$$

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi^0, \quad \lim_{t\uparrow\infty} \beta^t \overline{v}(x_t) = 0. \tag{3.5}$$

(iii) For any $x_0 \in X^*$, $\Pi^0(x_0) \neq \emptyset$. (iv) There exists $\hat{v} \in C^*(X)$ such that

$$\overline{v} < \hat{v}, \quad B\hat{v} < \hat{v}, \tag{3.6}$$

$$\exists \epsilon > 0, \quad \sup_{x \in X^*: \|x\| < \epsilon} \frac{\underline{v}(x) - \hat{v}(x)}{\overline{v}(x) - \hat{v}(x)} < \infty, \tag{3.7}$$

$$\exists \delta > 0, \quad \sup_{x \in X^*: \|x\| < \delta} \frac{\overline{v}(x) - \hat{v}(x)}{(B\hat{v})(x) - \hat{v}(x)} < \infty, \tag{3.8}$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^m$.

DP5. There exists an increasing sequence $\{K_j\}_{j=1}^\infty$ of compact subsets of X such that $\Gamma(K_j) \subset K_j$ for each $j \in \mathbb{N}$, and such that for any compact set $K \subset X$, there exists $j \in \mathbb{N}$ with $K \subset K_j$.

Theorem 3.1. (Martins-da-Rocha and Vailakis, 2010, Theorem 3.1) *Under DP0–DP5, the following conclusions hold:*
(a) *The Bellman operator B has a unique fixed point in $[\underline{v}, \overline{v}] \cap C^*(X)$.*
(b) *The unique fixed point of B in $[\underline{v}, \overline{v}] \cap C^*(X)$ is the value function v^*.*
(c) *For any $v \in [\underline{v}, \overline{v}] \cap C^*(X)$, the sequence $\{B^n v\}_{n=1}^\infty$ converges to v^* in the topology generated by the family $\{d_j\}_{j=1}^\infty$ of semidistances defined for all $f, g \in [\underline{v}, \overline{v}] \cap C^*(X)$ by*

$$d_j(f,g) = \sup_{x \in K_j \setminus \{0\}} \left| \ln \frac{f(x) - \hat{v}(x)}{\overline{v}(x) - \hat{v}(x)} - \ln \frac{g(x) - \hat{v}(x)}{\overline{v}(x) - \hat{v}(x)} \right|, \qquad j \in \mathbb{N}. \tag{3.9}$$

Let us compare the conclusions of Theorem 2.1 with those of Theorem 3.1 under DP0–DP5 (which imply the hypotheses of Theorem 2.1 by Corollary 3.1 below). In terms of existence of a fixed point, Theorem 2.1 is weaker than Theorem 3.1 in the sense that the latter shows that the fixed point lies in $C^*(X)$ (and thus the value function v^* is continuous). In terms of uniqueness, Theorem 2.1 is stronger than Theorem 3.1, which ensures uniqueness only in $[\underline{v}, \overline{v}] \cap C^*(X)$. In terms of convergence, Theorem 2.1 is considerably weaker than Theorem 3.1, which shows that convergence to v^* occurs from any initial function in $[\underline{v}, \overline{v}] \cap C^*(X)$ under a criterion much stronger than ours.

Let us now clarify which parts of DP0–DP5 are needed for the conclusions of Theorem 2.1. The following is a corollary of Theorem 2.1.

Corollary 3.1. *Assume DP0(i), DP0(ii), DP4(i), (3.4), and DP5. Then the functions $\underline{v}$ and $\overline{v}$ given by DP4(i) satisfy (2.12)–(2.16), and thus the conclusions of Theorem 2.1 hold.*

Proof. DP4(i) ensures (2.12)–(2.14). Condition (3.4) implies (2.15). To see (2.16), let $\{x_t\}_{t=0}^{\infty} \in \Pi$. By DP5, there exists a compact set K_j with $j \in \mathbb{N}$ such that $x_0 \in K_j$ and $\Gamma(K_j) \subset K_j$. Since $\overline{v}$ is continuous by DP4(i), we have $\overline{v}(x_t) \leq \max_{x \in K_j} \overline{v}(x) < \infty$ for all $t \in \mathbb{Z}_+$, which together with DP0(ii) implies (2.16). $\square$

The above result shows that a small subset of the conditions stated in DP0–DP5 is sufficient to show that the value function v^* is the unique fixed point of the Bellman operator B in $[\underline{v}, \overline{v}]$ and that convergence to v^* occurs from $\underline{v}$. In the proof, (2.16) is shown as a consequence of DP5; note that (3.5) is not assumed. The only role of DP0(i) is to keep $C^*(X)$ well-defined, while that of DP5 is to ensure (2.16). Thus DP0(i) can be dropped entirely if we do not require $\underline{v}, \overline{v} \in C^*(X)$, while DP5 can be replaced by a weaker sufficient condition for (2.16). The following result can be shown by modifying the proof of Corollary 3.1.

Corollary 3.2. *Let $\beta \in [0, 1)$. Suppose that there exist $\underline{v}, \overline{v} \in V$ satisfying (2.12)–(2.15). Suppose further that for any $x \in X$, there exists a set $K \subset X$ such that $\Gamma(K) \subset K$ and $\sup_{x \in K} \overline{v}(x) < \infty$. Then the conclusions of Theorem 2.1 hold.*

Appendix A Proof of Theorem 2.1

The proof consists of three lemmas and a concluding argument. The first lemma is similar to some of the results presented by Stokey and Lucas (1989, Theorem 4.3, Exercise 4.3):

Lemma A.1. *Let $\overline{v} \in V$ satisfy (2.16). Let $v \in V$ be a fixed point of B with $v \leq \overline{v}$. Then $v \leq v^*$.*

Proof. Let $x_0 \in X$. If $v(x_0) = -\infty$, then $v(x_0) \leq v^*(x_0)$. Consider the case $v(x_0) > -\infty$. Let $\epsilon > 0$. Let $\{\epsilon_t\}_{t=0}^{\infty} \subset (0, \infty)$ be such that $\sum_{t=0}^{\infty} \beta^t \epsilon_t \leq \epsilon$. Since $v = Bv$, for any $t \in \mathbb{Z}_+$ and $x_t \in X$, there exists $x_{t+1} \in \Gamma(x_t)$ such that

(A.1) $$v(x_t) \leq u(x_t, x_{t+1}) + \beta v(x_{t+1}) + \epsilon_t.$$

We pick $x_1 \in \Gamma(x_0), x_2 \in \Gamma(x_1), \ldots$ so that (A.1) holds for all $t \in \mathbb{Z}_+$. Then

$\{x_t\}_{t=1}^\infty \in \Pi(x_0)$. We have

$$v(x_0) \le u(x_0, x_1) + \beta v(x_1) + \epsilon_0 \tag{A.2}$$
$$\le u(x_0, x_1) + \beta[u(x_1, x_2) + \beta v(x_2) + \epsilon_1] + \epsilon_0 \tag{A.3}$$
$$\vdots \tag{A.4}$$
$$\le \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) + \epsilon, \quad \forall T \in \mathbb{N}. \tag{A.5}$$

Since $v(x_0) > -\infty$, we have $\beta^T v(x_T) > -\infty$ for all $T \in \mathbb{N}$. It follows that

$$v(x_0) - \epsilon - \beta^T v(x_T) \le \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}). \tag{A.6}$$

Applying $\underline{\lim}_{T\uparrow\infty}$ to both sides, we have

$$v(x_0) - \epsilon - \overline{\lim_{T\uparrow\infty}}\, \beta^T v(x_T) \le \underline{\lim_{T\uparrow\infty}} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) \le S(\{x_t\}_{t=0}^\infty). \tag{A.7}$$

By (2.16) we have $v(x_0) - \epsilon \le S(\{x_t\}_{t=0}^\infty) \le v^*(x_0)$. Since this is true for any $\epsilon > 0$, we have $v(x_0) \le v^*(x_0)$. Since x_0 was arbitrary, we obtain $v \le v^*$. $\square$

For any $v \in V$, provided that $B^n v \in V$ for all $n \in \mathbb{N}$, we define

$$v_n = B^n v, \qquad n \in \mathbb{N}. \tag{A.8}$$

The following remark follows from (2.10).

Remark A.1. Let $v, w \in V$ satisfy $v \le w$ and $Bw \le w$. Then $v_n \le w$ for all $n \in \mathbb{N}$.

Lemma A.2. *Let $\overline{v} \in V$ satisfy (2.14). Let $v \in V$ satisfy $v \le \overline{v}$. Then for any $T \in \mathbb{N}$, we have*

$$\forall x_0 \in X, \quad v_T(x_0) = \sup_{\{x_t\}_{t=1}^\infty \in \Pi(x_0)} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) \right\}. \tag{A.9}$$

Proof. Note from Remark A.1 that $v_t \le \overline{v}$ for all $t \in \mathbb{N}$. For any $x_0 \in X$, we have

$$v_1(x_0) = \sup_{x_1 \in \Gamma(x_0)} \{u(x_0, x_1) + \beta v(x_1)\} \tag{A.10}$$

$$= \sup_{x_1 \in \Gamma(x_0)} \sup_{\{x_t\}_{t=2}^{\infty} \in \Pi(x_1)} \{u(x_0, x_1) + \beta v(x_1)\} \tag{A.11}$$

$$= \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \{u(x_0, x_1) + \beta v(x_1)\}, \tag{A.12}$$

where (A.11) holds since $\{u(x_0, x_1) + \beta v(x_1)\}$ is independent of $\{x_t\}_{t=2}^{\infty}$,[6] and (A.12) follows by combining the two suprema (e.g., Kamihigashi, 2008, Lemma 1). It follows that (A.9) holds for $T = 1$.

Now assume (A.9) for $T = n \in \mathbb{N}$. For any $x_0 \in X$, we have

$$v_{n+1}(x_0) = \sup_{x_1 \in \Gamma(x_0)} \{u(x_0, x_1) + \beta v_n(x_1)\} \tag{A.13}$$

$$= \sup_{x_1 \in \Gamma(x_0)} \Bigg\{ u(x_0, x_1) \tag{A.14}$$

$$+ \beta \sup_{\{x_{i+1}\}_{i=1}^{\infty} \in \Pi(x_1)} \left\{ \sum_{i=0}^{n-1} \beta^i u(x_{i+1}, x_{i+2}) + \beta^n v(x_{n+1}) \right\} \Bigg\}$$

$$= \sup_{x_1 \in \Gamma(x_0)} \sup_{\{x_{i+1}\}_{i=1}^{\infty} \in \Pi(x_1)} \left\{ \sum_{t=0}^{n} \beta^t u(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) \right\} \tag{A.15}$$

$$= \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \left\{ \sum_{t=0}^{n} \beta^t u(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) \right\}, \tag{A.16}$$

where (A.14) uses (A.9) for $T = n$, (A.15) holds since $u(x_0, x_1)$ is independent of $\{x_{i+1}\}_{i=1}^{\infty}$, and (A.16) follows by combining the two suprema (e.g., Kamihigashi, 2008, Lemma 1). It follows that (A.9) holds for $T = n + 1$. By induction, (A.9) holds for all $T \in \mathbb{N}$. □

Lemma A.3. *Let* $\underline{v}, \overline{v} \in V$ *satisfy (2.12)–(2.15). Then* $\underline{v}^* \equiv \lim_{n\uparrow\infty} \underline{v}_n \ge v^*$.

Proof. Note from (2.12)–(2.14), (2.10), and Remark A.1 that $\{\underline{v}_t\}_{t=1}^{\infty}$ is in-

[6] This step requires our assumption that Γ is nonempty-valued.

creasing and bounded above by $\overline{v}$. Thus for any $x_0 \in X$, we have

$$\begin{aligned}
&\text{(A.17)} && \underline{v}^*(x_0) = \sup_{T\in\mathbb{N}} \underline{v}_T(x_0) \\
&\text{(A.18)} && \qquad = \sup_{T\in\mathbb{N}} \sup_{\{x_t\}_{t=1}^\infty \in \Pi(x_0)} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T) \right\} \\
&\text{(A.19)} && \qquad = \sup_{\{x_t\}_{t=1}^\infty \in \Pi(x_0)} \sup_{T\in\mathbb{N}} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T) \right\} \\
&\text{(A.20)} && \qquad \geq \sup_{\{x_t\}_{t=1}^\infty \in \Pi(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T) \right\} \\
&\text{(A.21)} && \qquad \geq \sup_{\{x_t\}_{t=1}^\infty \in \Pi(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) = v^*(x_0),
\end{aligned}$$

where (A.18) uses Lemma A.2, (A.19) follows by interchanging the two suprema (e.g., Kamihigashi, 2008, Lemma 1), and the inequality in (A.21) follows from the properties of $\underline{\lim}$ and $\overline{\lim}$.[7] It follows that $\underline{v}^* \geq v^*$. □

To complete the proof of Theorem 2.1, suppose that there exist $\underline{v}, \overline{v} \in V$ satisfying (2.12)–(2.16). Note that $[\underline{v}, \overline{v}]$ is a partially ordered set with the partial order $\leq$. Since B is a monotone operator, and since $B([\underline{v}, \overline{v}]) \subset [\underline{v}, \overline{v}]$ by (2.12)–(2.14) and (2.10), B has a fixed point v in $[\underline{v}, \overline{v}]$ by the Knaster-Tarski fixed point theorem (e.g., Aliprantis and Boder, 1999, p. 15). Since $\underline{v} \leq v$, we have $\underline{v}_n \leq v$ for all $n \in \mathbb{N}$ by Remark A.1. Thus $\underline{v}^* \leq v$. Since $v \leq v^*$ by Lemma A.1, and since $v^* \leq \underline{v}^*$ by Lemma A.3, it follows that $v \leq v^* \leq \underline{v}^* \leq v$. Hence $v = \underline{v}^* = v^*$. Therefore v^* is a unique fixed point, establishing (a) and (b); (c) holds since $\underline{v}^* = v^*$.

Appendix B Examples

B.1 Multiple Fixed Points

The Bellman operator B can have multiple fixed points in $[\underline{v}, \overline{v}]$ if (2.15) and (2.16) are violated. To illustrate this point, consider the trivial example

[7] We have $\underline{\lim}(a_t + b_t) \geq \underline{\lim}\, a_t + \underline{\lim}\, b_t$ and $\overline{\lim}(a_t + b_t) \geq \overline{\lim}\, a_t + \underline{\lim}\, b_t$ for any sequences $\{a_t\}$ and $\{b_t\}$ in $[-\infty, \infty)$ whenever both sides are well-defined (e.g., Michel, 1990, p. 706).

specified as follows:

$$\text{(B.1)} \qquad X = \mathbb{Z}_+, \quad \beta > 0; \qquad \forall i \in X, \quad \Gamma(i) = \{i+1\}, \quad u(i, i+1) = 0.$$

At each state $i \in X$, there is only one feasible choice $(i+1)$ with a return of zero. Thus $v^*(i) = 0$ for all $i \in X$. Let $\alpha > 0$. Define $\underline{v}, \overline{v} \in V$ by $\underline{v}(i) = -\alpha\beta^{-i}$ and $\overline{v}(i) = \alpha\beta^{-i}$ for all $i \in X$. We have $\underline{v} < \overline{v}$, which implies (2.12). Since $\underline{v}(i) = \beta\underline{v}(i+1)$ and $\overline{v}(i) = \beta\overline{v}(i+1)$ for all $i \in X$, (2.13) and (2.14) hold with equality. This observation alone shows that B has multiple fixed points in $[\underline{v}, \overline{v}]$. In fact, for any $a \in [-\alpha, \alpha]$, the function v defined by $v(i) = a\beta^{-i}$ for all $i \in X$ is a fixed point of B. Therefore B has a continuum of fixed points here. Note that the only feasible path from state 0 is given by $\{x_t\}_{t=1}^\infty = \{t\}_{t=1}^\infty$, and that $\beta^t\underline{v}(x_t) = -\alpha$ and $\beta^t\overline{v}(x_t) = \alpha$ for all $t \in \mathbb{Z}_+$. Thus (2.15) and (2.16) are violated in this example.

B.2 Nonconvergence to v^*

Even under (2.12)–(2.16), the sequence $\{\overline{v}_n\}_{n=1}^\infty$ (recall (A.8)) may not converge to v^*. To illustrate this point, consider the following example:

$$\text{(B.2)} \qquad X = \{(i,j) : i, j \in \mathbb{Z}_+, j \leq i\},$$

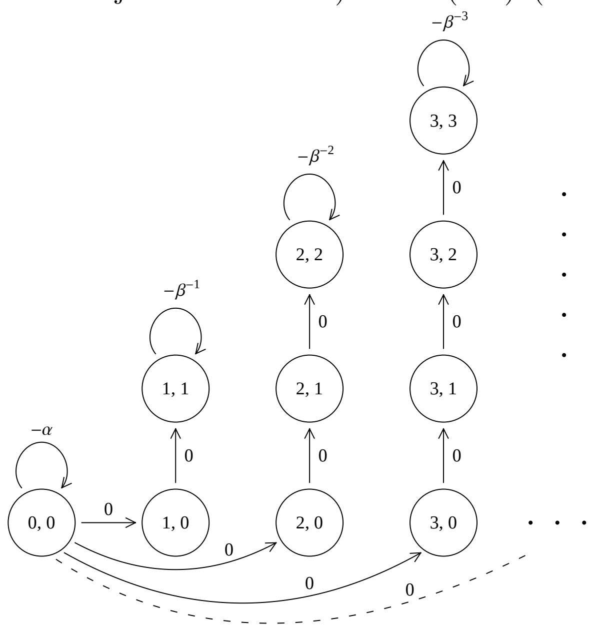
$$\text{(B.3)} \qquad \Gamma((i,j)) = \begin{cases} \{(i', 0) : i' \in \mathbb{Z}_+\} & \text{if } (i,j) = (0,0), \\ \{(i,j)\} & \text{if } i = j \neq 0, \\ \{(i, j+1)\} & \text{if } j < i, \end{cases}$$

$$\text{(B.4)} \qquad u((i,j),(i',j')) = \begin{cases} -\alpha & \text{if } (i,j) = (i',j') = (0,0), \\ -\beta^{-i} & \text{if } (i,j) = (i',j') \neq (0,0), \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \beta < 1$. See Figure 1.

The value function v^* can be easily computed as follows. Let $i \in \mathbb{N}$. Then at state (i,i), we have $v^*((i,i)) = -\beta^{-i}/(1-\beta)$. Note that $v^*((i, i-k)) = \beta^k v^*((i,i))$ for $k = 1, \ldots, i$; thus $v^*((i,j)) = -\beta^{i-j}v^*((i,i)) = -\beta^{-j}/(1-\beta)$. We have computed $v^*((i,j))$ for all $(i,j) \in X$ except $(i,j) = (0,0)$. If $x_t = (0,0)$ for all $t \in \mathbb{Z}_+$, then $S(\{x_t\}_{t=0}^\infty) = -\alpha/(1-\beta)$. If $x_1 = (i, 0)$ with $i > 0$, then $S(\{x_t\}_{t=0}^\infty) = \beta v^*((i,0)) = -\beta/(1-\beta) < -\alpha/(1-\beta)$. Hence it is never optimal to leave state $(0,0)$, so that $v^*((0,0)) = -\alpha/(1-\beta)$. To

Figure 1: States $(i,j) \in X$ (circles), feasible transitions (arrows), and associated returns (values adjacent to arrows) under (B.2)–(B.4)



summarize, we have

$$\text{(B.5)} \qquad v^*((i,j)) = \begin{cases} -\alpha/(1-\beta) & \text{if } (i,j) = (0,0), \\ -\beta^{-j}/(1-\beta) & \text{otherwise.} \end{cases}$$

Let $\underline{v} = v^*$ and $\overline{v} = 0$. Then $\underline{v} < \overline{v}$ and $B\underline{v} = \underline{v}$. Since $u \leq 0$, we have $B\overline{v} \leq \overline{v}$. Thus (2.12)–(2.14) hold. Since any feasible path eventually becomes constant, (2.15) holds; (2.16) is trivial. It follows that Theorem 2.1 applies.

Consider the decreasing sequence $\{\overline{v}_n\}_{n=1}^\infty$. If $(i,j) \neq (0,0)$, there is only one feasible transition from (i,j), so that $\overline{v}_n((i,j))$ can be directly computed:

$$\text{(B.6)} \qquad \overline{v}_n((i,j)) = \begin{cases} -\beta^{-j} \sum_{k=0}^{n-(i-j)-1} \beta^k & \text{if } i > 0 \text{ and } n \geq i - j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

This formula works for $(i,j) = (0,0)$ as well; i.e., $\overline{v}_n((0,0)) = 0$ for all $n \in \mathbb{N}$. To see this, define $\overline{v}_0 = \overline{v} = 0$. Then $\overline{v}_0((0,0)) = 0$. Let $n \in \mathbb{Z}_+$. With $\overline{v}_n$ given by (B.6), we have $\overline{v}_{n+1}((0,0)) = \beta \sup_{i \in X} \overline{v}_n((i,0)) = 0$ since $\overline{v}_n((i,0)) = 0$ for all $i \geq n$. By induction, $\overline{v}_n((0,0)) = 0$ for all $n \in \mathbb{N}$.

Now letting $\overline{v}^* = \lim_{n\uparrow\infty} \overline{v}_n$, we see that $\overline{v}^*((0,0)) = 0 > v^*((0,0))$ though $\overline{v}^*((i,j)) = v^*((i,j))$ for all $(i,j) \in X \setminus \{(0,0)\}$. It follows that $\overline{v}^* > v^*$; i.e., the sequence $\{\overline{v}_n\}_{n=1}^\infty$ fails to converge to v^*. On the other hand, the sequence $\{\overline{v}_n^*\}_{n=1}^\infty = \{B^n \overline{v}^*\}_{t=1}^\infty$ restarted from $\overline{v}^*$ converges to v^*; indeed, $\overline{v}_n^*((0,0)) = -\alpha(1 + \beta + \cdots + \beta^{n-1}) \to v^*((0,0))$ as $n \uparrow \infty$.

References

Aliprantis, C.D., Border, K.C., 1999, *Infinite Dimensional Analysis: A Hitchhiker's Guide,* 2nd ed. Springer-Verlag, Berlin.

Kamihigashi, T., 2008, On the principle of optimality for nonstationary deterministic dynamic programming, *International Journal of Economic Theory* 4, 519–525.

Le Van, C., Morhaim, L, 2002, Optimal growth models with bounded or unbounded returns: a unifying approach, *Journal of Economic Theory* 105, 158–187.

Martins-da-Rocha, V.F., Vailakis, Y., 2008, Existence and uniqueness of a fixed point for local contractions, Economic Essays, Getulio Vargas Foundation.

Martins-da-Rocha, V.F., Vailakis, Y., 2010, Existence and uniqueness of a fixed point for local contractions, *Econometrica* 78, 1127–1141.

Michel, P., 1990, Some clarifications on the transversality condition, *Econometrica* 58, 705–723.

Rinćon-Zapatero, J.P., Rodríguez-Palmero, C., 2003, Existence and uniqueness of solutions to the Bellman equation in the unbounded case, *Econometrica* 71, 1519–1555.

Rinćon-Zapatero, J.P., Rodríguez-Palmero, C., 2007, Recursive utility with unbounded aggregators, *Economic Theory* 33, 381–391.

Rinćon-Zapatero, J.P., Rodríguez-Palmero, C., 2009, Corrigendum to "Existence and uniqueness of solutions to the Bellman equation in the unbounded case" Econometrica, Vol. 71, No. 5 (September, 2003), 1519–1555, *Econometrica* 77, 317–318.

Stokey, N., Lucas, R.E., Jr., 1989, *Recursive Methods in Economic Dynamics.* Harvard University Press, Cambridge, MA.